Mixed Boundary Conditions in Heat Conduction

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Introduction

In mixed boundary value (MBV) problems, the nature of the boundary condition can change along a particular boundary (finite, semi-infinite or infinite in length), say from a Dirichlet condition to a Neumann condition. Most MBV problems are solved using classical techniques such as separation of variables (domain of limited extent) or transform methods (domain of semi-infinite or infinite extent) which lead to dual or triple integral equations. Also, they are usually solved when a steady state condition is reached [1]. In authors' knowledge, the only exception is the paper by Sadhal about solids with partially contacting interface [2]. In this work we deal with both steady state and transient MBV problems which are solved as inverse heat conduction (IHC) problems [3] using Green's functions [4] and superposition in space and time.

Problem formulation

Consider a plate of thickness L along x and semi-infinite in the $y-$direction with temperature-independent properties and initially at zero temperature. The plate is thermally insulated at $y=0$ and subject to a constant heat flux q_0 at its boundary surface $x=0$ for $0 \le y \le W_0$. The remaining part of this boundary is kept at zero temperature as well as the opposite boundary at $x=L$. A schematic of this two-dimensional problem denoted by X(21)1B(10)0Y20B0T0 is given in Fig. 1a, where "(21)" denotes mixed boundary conditions of 2nd and 1st kinds.

Its mathematical formulation is

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} = \frac{1}{\alpha}\frac{\partial T}{\partial t} \tag{1a}$$

$$-k\left(\frac{\partial T}{\partial x}\right)_{x=0} + hT = \begin{cases} q_0 & \text{for } 0 \le y \le W_0 \quad \text{with } h=0 \\ 0 & \text{for } y \ge W_0 \quad \text{with } h \to \infty \end{cases} \qquad T(L,y,t)=0 \tag{1b}$$

$$\left(\frac{\partial T}{\partial y}\right)_{y=0} = 0 \qquad T(x, y \to \infty, t) = 0 \tag{1c}$$

$$T(x,y,0)=0 \tag{1d}$$

We are interested in computing the temperature distribution of the plate. For this purpose, the problem can be solved for the unknown heat flux $q_x(0,y,t) = q_x(y,t)$ at the boundary surface $x=0$ for $y \ge W_0$ (flux-based formulation) or, alternatively, for the unknown temperature $T(0,y,t)=T(y,t)$ at the boundary surface $x=0$ for $0 \le y \le W_0$ (temperature-based formulation).

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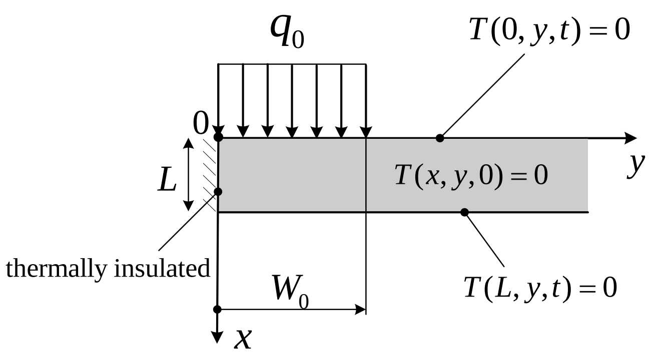

Figure 1 – A schematic of the two-dimensional transient problem with mixed boundary conditions at the boundary surface $x=0$.

In both methods (which are in part numerical and in part analytical), we have to solve a 2D inverse heat conduction problem (IHCP) [4]. In the former, we have to determine the unknown heat flux at the outer boundary surface $x=0$. In the latter, we have to derive the temperature at the same boundary surface. In this work we will deal only with the flux-based formulation for which there are no measurement errors as the temperatures are exactly equal to zero at $x=0$ for $y \geq W_0$.

Solution of the inverse problem

This inverse formulation requires that we first solve the forward problem denoted by X21B(*y*-*t*-)0 Y20B0T0, where "(*y*-)" denotes an arbitrary space function along *y*. For this problem, the discontinuous mixed boundary condition in Eq. (1b) reduces to a simple non-homogeneous boundary condition of the 2nd kind, where the heat flux $q_x(y,t)$ in the $x-$direction is unknown for $y \geq W_0$. It will simply be termed as $q(y,t)$ afterwards.

As the above heat flux is not uniform, it can be convenient to perform a discretization in space with uniformly-spaced elements (Δy). The *j*-th heat flux component $q_j^{(n)}$ (with $j=1,2,3,\ldots$) at the time step $n\Delta t$ is applied to the domain $y \in [W_0+(j-1)\Delta y, W_0+j\Delta y]$ where it is assumed to be uniform. In a matrix form we have $\tilde{\mathbf{q}} = -[\Delta\mathbf{X}]^{-1}\mathbf{X}_0$, where

$$\tilde{\mathbf{q}} = \begin{bmatrix} \tilde{\mathbf{q}}^{(1)} \\ \tilde{\mathbf{q}}^{(2)} \\ \tilde{\mathbf{q}}^{(3)} \\ \ldots \end{bmatrix} \qquad \Delta\mathbf{X} = \begin{bmatrix} \Delta\mathbf{X}^{(1)} & 0 & 0 & \ldots \\ \Delta\mathbf{X}^{(2)} & \Delta\mathbf{X}^{(1)} & 0 & \ldots \\ \Delta\mathbf{X}^{(3)} & \Delta\mathbf{X}^{(2)} & \Delta\mathbf{X}^{(1)} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{bmatrix} \qquad \mathbf{X}_0 = \begin{bmatrix} \mathbf{X}_0^{(1)} \\ \mathbf{X}_0^{(2)} \\ \mathbf{X}_0^{(3)} \\ \ldots \end{bmatrix} \tag{2}$$

If we consider a generic time step *n*, we have

$$\tilde{\mathbf{q}}^{(n)} = \begin{bmatrix} \tilde{q}_1^{(n)} \\ \tilde{q}_2^{(n)} \\ \tilde{q}_3^{(n)} \\ \ldots \end{bmatrix} \qquad \Delta\mathbf{X}^{(n)} = \begin{bmatrix} \Delta X_{11}^{(n)} & \Delta X_{12}^{(n)} & \Delta X_{13}^{(n)} & \ldots \\ \Delta X_{21}^{(n)} & \Delta X_{22}^{(n)} & \Delta X_{23}^{(n)} & \ldots \\ \Delta X_{31}^{(n)} & \Delta X_{32}^{(n)} & \Delta X_{33}^{(n)} & \ldots \\ \ldots & \ldots & \ldots & \ldots \end{bmatrix} \qquad \mathbf{X}_0^{(n)} = \begin{bmatrix} X_{10}^{(n)} \\ X_{20}^{(n)} \\ X_{30}^{(n)} \\ \ldots \end{bmatrix} \tag{3}$$

where $\Delta X_{ij}^{(n)}$ is the dimensionless temperature due to the *j*-th dimensionless heat flux component at the surface locations $\tilde{y} = 1+(i-1/2)\Delta\tilde{y}$ ($i=\mathbf{1},\mathbf{2},\ldots$) and at the *n*-th time step ($\tilde{q}_j^{(n)} = q_j^{(n)}/q_0$). By applying a superposition in space and time, the above temperatures may be computed as solutions of the basic "building block" X21B(*y*5)0Y20B0T0. These solutions can be derived by using Green's functions approach [4]. Note that the $\Delta\mathbf{X}^{(n)}$ matrix is diagonal as $\Delta X_{11}^{(n)} = \ldots = \Delta X_{ii}^{(n)}$ and also symmetric with respect to it as $\Delta X_{ij}^{(n)} = \Delta X_{ji}^{(n)}$. Similarly, $X_{i0}^{(n)}$ is the dimensionless temperature due to the 0-th heat flux component (applied to the domain $y \in [0, W_0]$) at the surface locations $\tilde{y} = 1 + (i - 1/2)\Delta\tilde{y}$ ($i = \mathbf{1}, \mathbf{2}, \ldots$) and at the *n*-th time step. Then, by using the concept of penetration distance [5], we can consider a limited number of heat flux components depending upon the accuracy desired.

Results and conclusions

For a uniform heat flux applied to the region $0 \to W_0 / L = 2$, the results are given in Fig. 2 versus y/L for different values of the dimensionless time defined as $\alpha t / L^2$. In detail, $\Delta y / L = 0.05$ and $\Delta(\alpha t / L^2) = 0.1$. The known heat flux applied between 0 and $W_0 / L = 2$ is also plotted in Fig. 2 for a better comparison with the unknown heat flux computed. Note that the dimensionless steady-state heat flux is of about -3.85 near the q/T interface $W_0 / L = 2$ vs. the expected value of $-\infty$. This is due to the fact that -3.85 is the average value of the heat flux between 2 and 2.05. Reducing $\Delta y / L$ gives a higher value of the negative heat flux.

Once the heat flux components are computed, it is possible to calculate the dimensionless heat rate that exits the boundary at $\tilde{x} = 0$ for $y \geq W_0 / L$ when a steady state is reached. It is given by $\tilde{q}_{ex}^{(0)} = -0.4315$. As the dimensionless heat rate that enters the boundary at $\tilde{x} = 0$ for $y \in [0, W_0]$ is $\tilde{q}_{en}^{(0)} = 2$, the heat rate exiting the back side of the plate kept at zero temperature is of 1.5685.

Note that the heat flux computation can be made faster since it is not necessary to have uniformly-spaced elements. Near the q/T interface $W_0 / L = 2$, in fact, we need smaller elements and larger ones can be used further away.

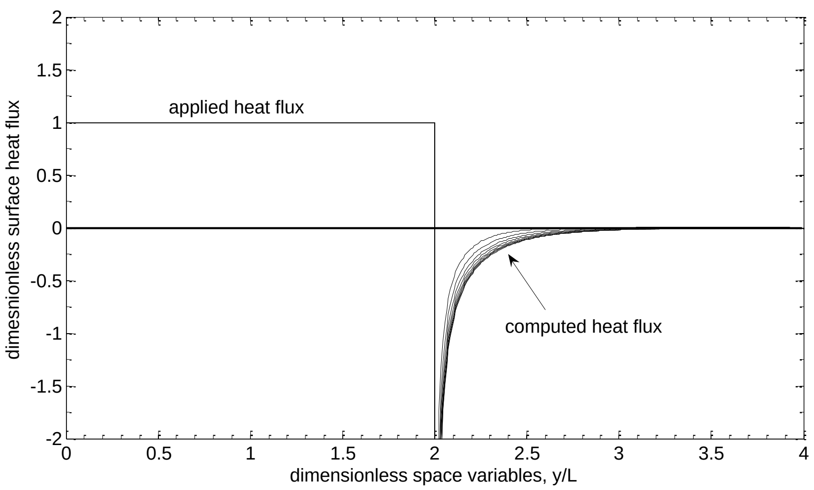

Figure 2 – Surface heat flux as a function of y/L with $\alpha t / L^2$ as a parameter. Ten time steps (n = 10) have been considered.

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